Chapter 2: Analytic Functions

Definition A function $f : \Omega \longrightarrow \mathbb{C}$ is a rule that assigns to each $z \in \Omega \subseteq \mathbb{C}$ a unique complex number $f(z) \in \mathbb{C}$. The set Ω is the domain of definition of f. If $S \subseteq \Omega$, then the set

$$f(S) \overset{\text{def}}{=} \{ f(z) : z \in S \}$$

is the image of S. The set $f(\Omega)$ is the range of f. Points in $f(\Omega)$ are called values of f. //

If $f : \Omega \longrightarrow \mathbb{C}$ is a function, then the value $f(x+iy) = u + iv$ depends on $(x,y) \in \mathbb{R}^2$. Collecting all values, we decompose f into real and imaginary parts:

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

where $u, v : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are real valued functions of 2 real varibles.

$$\operatorname{Re} f = u \quad \text{and} \quad \operatorname{Im} f = v. \qquad //$$

Example Some examples of functions.

(1) $f(z) = z^2$. Let $z = x+iy$. Then

$$f(z) = (x+iy)(x+iy) = x^2 - y^2 + i(2xy).$$

So, $u(x,y) = x^2 + y^2$ and $v(x,y) = 2xy$.

(2) $f(z) = |z|^2 = x^2 + y^2$ so $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$.

Such a function is real-valued.

(3) Polynomials are functions of the form

$$P(z) = a_0 + a_1 z + \dots + a_n z^n$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$. Then the degree of P is n.

(4) Rational functions are functions of the form $P(z)/Q(z)$ where $P(z), Q(z)$ are polynomials. The domain of definition is whereever $Q(z) \neq 0$.

(5) If we use polar coordinates for z, then a function f can be written

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta).$$

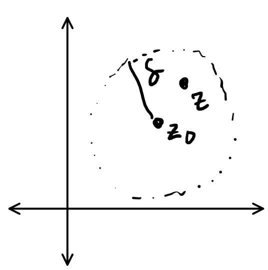
E.g. $f(z) = z^2$. If $z = re^{i\theta}$, then

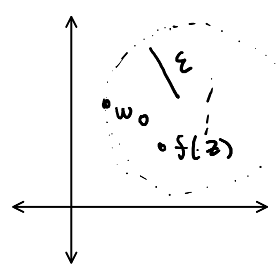
$$f(z) = r^2 e^{i2\theta} = \underbrace{r^2 \cos 2\theta}_{u(r,\theta)} + i\underbrace{r^2 \sin 2\theta}_{v(r,\theta)}.$$

(6) $f(z) = z^{1/n}$. This is not a function at all! We saw last time that $z^{1/n}$ has n distinct values. Such a "function" is called multiple-valued. We can make this into a single-valued function by assigning a single value of $z^{1/n}$ for each z. For instance, taking the principal nth root of z. //

Limits of Functions

Definition (Limit of a function) Suppose f is defined in a deleted neighborhood of $z_0 \in \mathbb{C}$. We say that the limit of f as z approaches z_0 is $w_0 \in \mathbb{C}$ if: for all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |z - z_0| < \delta$ implies $|f(z) - w_0| < \varepsilon$.





In this case we write

$$\lim_{z \to z_0} f(z) = w_0.$$

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Intuitively, the limit of f at z_0 is w_0 if we can make $f(z)$ arbitrarily close to w_0 by taking z sufficiently close to z_0.

$\boxed{\text{Example}}$ We show that $\lim_{z \to i} z^2 = -1$ using the definition.

Proof. Let $\varepsilon > 0$. Note $|z^2 - (-1)| = |z-i||z+i|$. First suppose $0 < |z-i| < 1$. Then

$$\begin{aligned} |z+i| &= |z - i + 2i| \\ &\le |z-i| + |2i| \\ &= \underbrace{|z-i|}_{<1} + 2 < 3. \end{aligned}$$

Choose $\delta = \min\{\frac{\varepsilon}{3}, 1\}$. Then $0 < |z-i| < \delta$ implies

$$|z^2 - (-1)| = |z-i||z+i| < \frac{\varepsilon}{3} \cdot 3 = \varepsilon. \quad \blacksquare$$

$\boxed{\text{Theorem (Limits are Unique)}}$ If f has a limit at z_0, then it is unique.

Proof. Assume $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} f(z) = W_0$.

Let $\varepsilon > 0$. Choose $\delta_1 > 0$ such that

$$0 < |z - z_0| < \delta_1 \implies |f(z) - w_0| < \varepsilon/2.$$

Choose $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_2 \implies |f(z) - W_0| < \varepsilon/2.$$

Now, if $0 < |z - z_0| < \min\{\delta_1, \delta_2\}$, then

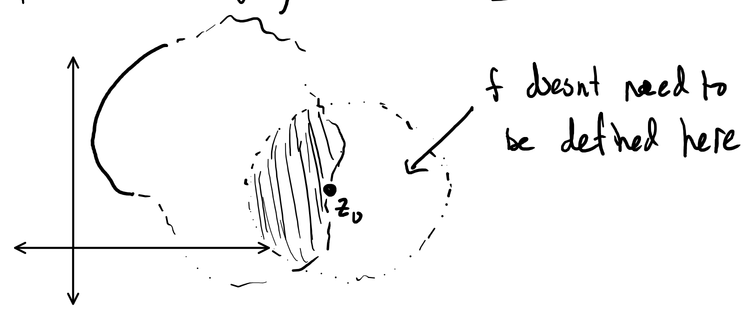
$$\begin{aligned} |w_0 - W_0| &= |w_0 - f(z) + f(z) - W_0| \\ &\le |w_0 - f(z)| + |f(z) - W_0| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary $|w_0 - W_0| = 0$. Hence, $w_0 = W_0$. $\blacksquare$

Limits can also be taken if z_0 is a boundary point of a region R on which f is defined. We simply require that the inequality

$$0 < |z - z_0| < \delta$$

be satisfied only for points belonging to R and a deleted neighborhood of z_0.



Uniqueness of limits can be used to show that a limit does not exist.

Example The function $f(z) = \dfrac{z}{\bar{z}}$ has no limit at 0.

Let $z = x + i$, then $f(z) = \dfrac{x+iy}{x-iy}$.

Along the real axis, $\operatorname{Im} z = 0$ so $z = x$. So $f(z) = \dfrac{x}{x} = 1$.
Along the imaginary axis, $\operatorname{Re} z = 0$ so $z = iy$. So $f(z) = \dfrac{iy}{-iy} = -1$.
Taking the limit along these axes, gives different values. By uniqueness of limits, the limit doesn't exist. //

Theorems on Limits

Theorem (Limits in Terms of Re f / Im f) Suppose that

$$f(z) = f(x+iy) = u(x,y) + iv(x,y).$$

Then

$$\lim_{x+iy \to x_0+iy_0} f(x+iy) = u_0 + iv_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 .$$

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < |x+iy - (x_0 + iy_0)| < \delta \quad \Longrightarrow \quad |u(x,y) + iv(x,y) - (u_0 + iv_0)| < \varepsilon .$$

Notice that $|x+iy - (x_0+iy_0)| = \underbrace{\sqrt{(x-x_0)^2 + (y-y_0)^2}}_{\text{distance between } (x,y),\,(x_0,y_0)}$

Notice also that

$$|u(x,y) - u_0| \le |u(x,y) + iv(x,y) - (u_0+iv_0)| < \varepsilon \quad \text{and}$$

$$|v(x,y) - v_0| \le |u(x,y) + iv(x,y) - (u_0+iv_0)| < \varepsilon .$$

This proves the claim.

($\Leftarrow$) See the book.

$\blacksquare$

$\boxed{\text{Theorem (Limit Laws)}}$ Suppose

$$\lim_{z\to z_0} f(z) = w_0 \quad \text{and} \quad \lim_{z\to z_0} F(z) = W_0 .$$

Then (1) $\displaystyle \lim_{z\to z_0} \big(f(z) + F(z)\big) = w_0 + W_0$

(2) $\displaystyle \lim_{z\to z_0} f(z)F(z) = w_0 W_0$

(3) $\displaystyle \lim_{z\to z_0} f(z)/F(z) = \frac{w_0}{W_0}$, if $W_0 \neq 0$.

Proof. Use preceding theorem together with limit laws from caclulus.

$\blacksquare$

$\boxed{\text{Example}}$ If $p(z)$ is a polynomial, then

$$\lim_{z\to z_0} p(z) = p(z_0).$$

Write $p(z) = a_0 + a_1 z + \cdots + a_n z^n$. By the limit laws:

$$\lim_{z \to z_0} p(z) \overset{(1)}{=} \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 z + \cdots + \lim_{z\to z_0} a_n z^n$$

$$\overset{(2)}{=} \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 \lim_{z\to z_0} z + \cdots + \lim_{z\to z_0} a_n \lim_{z\to z_0} z^n$$

$$= a_0 + a_1 \lim_{z\to z_0} z + \cdots + a_n \lim_{z\to z_n} z^n$$

Also $\lim_{z\to z_0} z^n = z_0^n$ by induction.

$$= a_0 + a_1 z_0 + \cdots + a_n z_0^n$$
$$= p(z_0).$$

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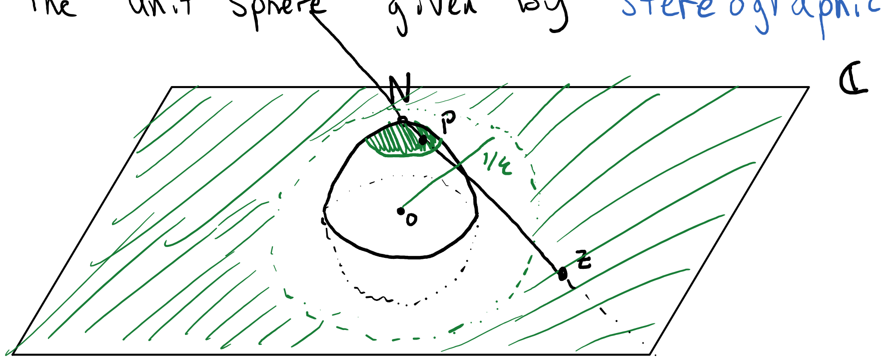
Definition (Extended Complex Plane / Riemann Sphere)

The extended complex plane is the set $\mathbb{C}$ together with a symbol ∞ called the point at infinity. There is a one-to-one correspondence between the extended complex plane and the unit sphere given by stereographic projection.



The point N corresponds to ∞ and any point P on the sphere corresponds to a unique point $z \in \mathbb{C}$ lying at the intersection of $\mathbb{C}$ and the line between P and N.

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Definition (Neighborhood of infinity) Let $\varepsilon > 0$. The set

$$\{ z \in \mathbb{C} : |z| > 1/\varepsilon \}$$

is called a neighborhood of ∞. Geometrically, a neighborhood of infinity is the exterior of a circle $C_{1/\varepsilon}(0)$, which corresponds to a neighborhood of N on the Riemann sphere. //

Meaning is easily given to limits

$$\lim_{z \to z_0} f(z) = w_0$$

(Limits involving ∞)

where z_0 or w_0 are allowed to be ∞. We simply replace the appropriate neighborhood in the original definition with neighborhoods of ∞.

$\boxed{\text{Theorem (Limits Involving Infinity)}}$

(1) If $\lim_{z \to z_0} 1/f(z) = 0$, then $\lim_{z \to z_0} f(z) = \infty$.

(2) If $\lim_{z \to 0} f(1/z) = w_0$, then $\lim_{z \to \infty} f(z) = w_0$.

(3) If $\lim_{z \to 0} 1/f(1/z) = 0$, then $\lim_{z \to \infty} f(z) = \infty$.

Proof.

(1) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < |z - z_0| < \delta \quad \text{implies} \quad \left| \frac{1}{f(z)} - 0 \right| < \varepsilon.$$

The condition $\left|\frac{1}{f(z)}\right| < \varepsilon$ is the same as $\frac{1}{\varepsilon} < |f(z)|$, i.e. $f(z)$ is in a neighborhood of ∞.

(2) Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$0 < |z - 0| < \delta \quad \text{implies} \quad |f(1/z) - w_0| < \varepsilon.$$

The condition $0 < |z| < \delta$ is equivalent to $\frac{1}{\delta} < \left|\frac{1}{z}\right|$. Replacing $1/z$ w/ z we get

$$1/\delta < |z| \quad \text{implies} \quad |f(z) - w_0| < \varepsilon. \qquad \blacksquare$$

$\boxed{\text{Example}}$ $\lim_{z \to \infty} \frac{2z^4+1}{z^3+1} = \infty$. Using (3), we compute

$$\lim_{z \to 0} \frac{(1/z)^3+1}{2(1/z)^4+1} = \lim_{z \to 0} \frac{1+z^3}{z^3} \Big/ 2\left(\frac{1+z^4}{z^4}\right) = \lim_{z \to 0} \frac{(1+z^3)\cdot z^4}{2(1+z^4)\, z^3}$$

$$= \lim_{z \to 0} \frac{(1+z^3)\, z}{2(1+z^4)} = 0. \quad //$$